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Solitons of q-deformed quantum lattices and the quantum soliton

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Abstract

We use the classical N-soliton solution of a q-deformed lattice, the Maxwell–Bloch (MB) lattice, which we reported recently (Rybin A V, Varzugin G G, Timonen J and Bullough R K 2001 *J. Phys. A: Math. Gen.* **34** 157) in order, ultimately, to fully comprehend the 'quantum soliton'. This object may be the source of a new information technology (Abram I 1999 Quantum solitons *Phys. World* 21–4). We suggested in Rybin *et al* 2001 that a natural quantum mechanical matrix element of the q-deformed *quantum* MB lattice becomes in a suitable limit the classical 1-soliton solution of the classical q-deformed MB lattice explicitly derived by a variant of the Darboux–Bäcklund method. The classical q-deformed MB lattice was introduced in Bogoliubov N M, Rybin A V, Bullough R K and Timonen J 1995 *Phys. Rev.* A **52** 1487. In this short paper the quantum inverse method is viewed as a Darboux–Bäcklund transformation at quantum level, *two* q-deformed quantum lattices are introduced and solved, and relevant matrix elements are formally derived. Further investigation of the classical limits of these matrix elements must however be deferred until future work.

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1. Introduction and background

In a short paper [1] we reported for the first time N-soliton solutions (and in detail derived especially the 1-soliton solution) of an integrable q-*deformed* one-dimensional *lattice* which we had already introduced in [3]; and the soliton solutions found depend explicitly on q

($q = e^{\gamma}$ and $\gamma > 0$ is a real parameter). It is a feature of integrable q-deformed lattices that q-deformation is possible *only* for such lattices; for in continuum limit the equations and their solutions no longer depend on q (or γ). Thus, in the case of the lattice introduced in [3] this reduces in continuum limit to, with $q \to 1$, the classical undeformed Maxwell–Bloch (MB) envelope equations, also called the SIT (for 'self-induced transparency') equations; and this is why we called the lattice the MB lattice. Since *lattices* are intrinsic to q-deformations, and our lattice is by construction integrable, some report on it is appropriate at a SIDE meeting; and for SIDE IV we report on aspects of the results obtained so far—particularly in the quantum cases of this lattice.

Our papers [3,4] followed up ideas of quantum groups and particularly the q-bosons [3–6], and the relevance of these to integrable systems theory. By working with the q-deformed *Lie–Poisson algebras* $su_q(2)$ we derived as a c-number (i.e. as a classical) equation both the q-deformed MB lattice system and its zero-curvature representation. The q-dependent soliton solutions (q-solitons) were then obtained by a variant of the Darboux–Bäcklund dressing method in the subsequent paper [1].

The hierarchy of continuum classical field equations 'reduced Maxwell–Bloch' (RMB) [8, 9], SIT (or envelope MB) [7, 9], sine-Gordon, and nonlinear Schrödinger (NLS) equations, each in 1+1 dimensions, are completely integrable Hamiltonian systems in the usual sense, and the hierarchy epitomizes the way in which, under a succession of slowly varying amplitude and phase approximations [10, 11] integrability is handed down in the fashion recently explored by Calogero [12]. For example, [10] (and see also reference [96] therein) shows the wide-ranging *physical application* of these equations—not just in quantum optics and nonlinear optics (to which [10] is mostly devoted) but also, through the *quantum* NLS model, to low-temperature physics and the theory of Bose–Einstein condensation in magnetic traps at some hundreds of nanokelvin—a topic of considerable theoretical [10, 11] and experimental [13] interest at this time, the latter [13] apparently justifying the theory [10, 11] for the *non-integrable quantum NLS* in *three* space dimensions.

The classical NLS models are also important to modern optical communications—especially trans-oceanic optical communication [10, 11]. In particular, the 'optical soliton', a solution of the classical NLS in 1 + 1 dimensions, provides the fundamental 'bit' for optical communication in fibres, trans-oceanic or not, and its quantum version, the 'quantum soliton' [2, 10, 11] may be an important realization of a 'qubit' for 'quantum information' (quantum computing, logic gates [2, 11]) while, compared with current realizations in terms of cavity quantum electrodynamics [11], this qubit will be relatively easily created, and then maintained *without dissipation*, experimentally. In [1] it was already suggested that 'quantum solitons' could be advantageously studied through the quantized form of the MB lattice [1, 3] (for an early paper on the quantum solitons of the related sine-Gordon system see [14]). The quantum q-deformed MB lattice is easily constructed by working with the *quantum group* $su_q(2)$, that is $U_q(su(2))$, rather then the Poisson–Lie algebra $su_q(2)$. In this paper for the SIDE IV meeting we elaborate further on this theme: we have noted that *lattices* are essential for q-deformation; q-deformation is a theme of SIDE IV (as is the physical applications which we have partly exemplified above); moreover at the classical level mostly studied in [1, 3] the MB lattice equations are indeed simply integrable differential-difference equations (in the dynamical variables s_n, H_n and β_n at each lattice side n)—a hallmark of the SIDE meetings. The corresponding quantum lattices, the quantum MB lattices, which are the main concern of this paper, replace these dynamical variables by quantum objects satisfying *quantum integrable* Heisenberg differential-difference equations, providing a natural focus of interest for SIDE IV.

These last remarks justify our study for readers of the SIDE IV proceedings. But ultimately it is the aim of the investigation to compute quantum mechanical matrix elements like

$$\lim_{N\to\infty} \langle 0|C(\lambda_1)C(\lambda_2)\ldots C(\lambda_{N-1})\beta_n B(\lambda_1)B(\lambda_2)\ldots B(\lambda_N)|0\rangle \tag{1}$$

where the $B(\lambda)$ are 'creation operators' acting on the vacuum $|0\rangle$ and the $C(\lambda) = B^\dagger(\lambda)$ are 'annihilation operators'; the operators β_n are viewed as 'electric field operators' for the quantum lattice in particular. We wish to compute matrix elements (1) because, for example, Wadati and others have shown [9, 10, 15, 16] that the classical soliton solution of the (1 + 1)-dimensional attractive NLS model (with negative coupling constant $c < 0$) derives from a quantum mechanical matrix element

$$\lim_{n\to\infty} \langle n, X, t|\hat{\phi}(x - vt)|n + 1, X', t\rangle \tag{2}$$

where $\hat{\phi}(x, t)$ satisfies (after scaling) the quantum attractive NLS equation (with $c < 0$)

$$-\mathrm{i}\hat{\phi}_t = \hat{\phi}_{xx} - 2c\hat{\phi}^\dagger\hat{\phi}\hat{\phi} \tag{3}$$

(where the notation is the usual $\hat{\phi}_t = \partial\hat{\phi}/\partial t$ etc) and the quantum fields $\hat{\phi}$, $\hat{\phi}^\dagger$ satisfy $[\hat{\phi}, \hat{\phi}^\dagger] = \delta(x - x')$ for bosons in 1 + 1 dimensions: v is a velocity and the states $|n + 1, X', t\rangle$ in (2) are wave packets deriving from states $|n + 1, P\rangle$ which are simultaneous eigenstates of $\hat{N} \equiv \int \hat{\phi}^\dagger\hat{\phi}\,\mathrm{d}x$, $\hat{P} \equiv -\mathrm{i}\int \hat{\phi}^\dagger\hat{\phi}_x\,\mathrm{d}x$ (and of a further infinity of mutually commuting operators which include the (attractive) Hamiltonian operator $\hat{H} = \int[\hat{\phi}^\dagger_x\hat{\phi}_x + c\hat{\phi}^\dagger\hat{\phi}^\dagger\hat{\phi}\hat{\phi}]\,\mathrm{d}x$, $c < 0$, thus establishing 'complete quantum integrability' [10, 11]). In principle we expect that the q-dependent 1-soliton solution we gave explicitly in [1] derives in a similar way through matrix elements like (1), although at the time of writing we have still to demonstrate these results.

The demonstration will be important because in the case of the NLS model in optical fibres one would like, for example, to 'squeeze' the 'photon number' n so that quantum fluctuations in n are reduced below the quantum-induced 'shot noise' level, thus increasing the precision of the 'bit' (which as a quantum object is actually a 'qubit') carrying information in the fibre. However, in the sequence for $n \to \infty$ in equation (2) n is fixed, essentially as an eigenvalue of the number operator $\hat{N}$, and cannot be squeezed. Even so, from different initial data, namely from initial *coherent states* of the electric field operator rather than $|n\rangle$ states, data provided in practice by an incident laser light source, such squeezing has been detected and followed [2, 10, 11, 17, 18]. This physics seems to mean that modified matrix elements like (2) will describe *at quantum level* the evolution of initially coherent incident light in the fibre into appropriate multisoliton solutions much as this evolution must take place in a classical theory.

Interesting and even important as this may be, we can do little more in this short paper than describe the quantum MB lattice and its eigenstates preparatory to evaluating the relevant matrix elements like (2) elsewhere.

The q-deformed 1-soliton solution (the q-soliton) at classical level *was* obtained in [1] as noted. For reference this is given with a sketch of the classical Darboux–Bäcklund method of solution as the early part of section 2. For the matrix element (2) Wadati [15] uses the coordinate Bethe ansatz [19] descriptions of the eigenstates $|n, P\rangle$ by starting from the 'n-string' bound states solutions of the quantum NLS model; section 2 finally arrives at the corresponding N-string bound state solutions of the quantum q-deformed lattice MB (LMB) equations which are introduced and then studied. Indeed, section 2 shows that there are at least *two* such q-deformed LMB equations, LMB I and LMB II with Hamiltonians $\boldsymbol{H}_{\mathrm{I}}$, $\boldsymbol{H}_{\mathrm{II}}$ respectively, which have N-string bound state solutions, and that this is a *generic* property which is important to a class of lattices and their various continuum limits. Their eigenstates we shall here take in the strict Bethe ansatz (or quantum inverse method) form rather than in the coordinate Bethe ansatz: these Bethe states are then the $B(\lambda_1)\ldots B(\lambda_N)|0\rangle$ states needed to evaluate the matrix elements taken in the explicit form (1) for, with N replacing $n + 1$, $N \to N - 1$. For the expression of these states in coordinate representation and the evaluation of the classical 1-soliton solutions for $N \to \infty$ following Wadati's work [15, 16] for the quantum NLS model we need the further work which must now be reported elsewhere. An interesting point is that only for each fixed n does the quantum NLS model have a stable ground state [9] (otherwise the system can lower its energy by sucking in (from the vacuum $|0\rangle$) more and more photons). This problem is avoided by relativistic co-variance as exemplified by the sine-Gordon system, and we believe that this property extends to the quantum SIT equations and thence to the quantum LMB (lattice) systems.

This suggests that the LMB systems are much the deeper systems for studying quantum systems of potential physical significance.

2. Two exactly solvable quantum models

In [3] we constructed the completely integrable *classical* lattice system whose equations of motion for three dynamical variables s_n, H_n and β_n at each lattice site n can be written as

$$\begin{aligned}
\partial_t \beta_n &= -\frac{1}{2} q^{2(N_n+H_n)}(\beta_{n+1}+\beta_n) - \frac{\mathrm{i}}{2} q^{2N_n}(s_n + s_{n-1}) \\
\partial_t s_n &= -\frac{\mathrm{i}}{2}(\beta_n+\beta_{n+1})(1+2\gamma s_n s_n^*) + \frac{1}{2} q^{2(N_n+H_n)}(s_n+s_{n-1}) \\
\partial_t H_n &= \frac{\mathrm{i}}{2}(s_n - \mathrm{i}q^{2H_n}\beta_n)(\beta_n^* + \beta_{n+1}^*) - \frac{\mathrm{i}}{2}(s_n^* + \mathrm{i}q^{2H_n}\beta_n^*)(\beta_n+\beta_{n+1}).
\end{aligned} \tag{4}$$

Here $q^{2N_n} = 1 + 2\gamma\beta_n^*\beta_n$, $q = \mathrm{e}^\gamma$ and $\gamma > 0$, is a real parameter (a coupling constant—see below). Reference to [3] shows that in equations (4) we use $s_n = \sqrt{2\gamma}\,\chi_n + \mathrm{i}q^{2H_n}\beta_n$: in [3] the second equation is for $\partial_t \chi_n$. As can be checked (and cf [3]) when the lattice spacing $\Delta \to 0$ for a continuum limit with

$$\begin{aligned}
&t \to t\Delta^{-1} \qquad x = n\Delta \qquad \beta_n = \sqrt{\Delta}\,\mathcal{E}(x) \qquad \chi_n = \Delta S(x) \qquad H_n = \Delta S^3(x) \\
&\gamma = \kappa\Delta/2 \qquad \kappa > 0
\end{aligned} \tag{5}$$

one reaches the resonant sharp-line form of the envelope MB (or SIT) equations (1) of [1] via the definitions

$$\varepsilon(\xi,\tau) = 2\mathcal{E}(x,t) \qquad \rho(\xi,\tau) = -2\mathrm{i}S(x,t) \qquad N(\xi,\tau) = 2S^3(x,t). \tag{6}$$

Our use of 'LMB equations' for equations (4) stems from this fact, as explained. A Hamiltonian for this LMB system is [3]

$$\mathcal{H}_c^L = \tfrac{1}{2}\sum_{n=1}^{M}\left\{\sqrt{2\gamma}\left[\chi_n^*(\beta_{n+1}+\beta_n) + \chi_n(\beta_{n+1}^*+\beta_n^*)\right] + \mathrm{i}q^{2H_n}\left(\beta_{n+1}^*\beta_n - \beta_{n+1}\beta_n^*\right)\right\}. \tag{7}$$

For $M < \infty$ it would be natural to impose periodic boundary conditions. But we shall look for lattice soliton solutions and here think of $M \to \infty$ with suitable boundary conditions still to be specified. The Poisson brackets of equation (7) are

$$\{X_n^*, X_m\} = \mathrm{i}\{2H_n\}\delta_{mn} \qquad \{H_n, X_m\} = -\mathrm{i}X_n\delta_{mn} \tag{8}$$

and the quantities X_n^*, X_n and H_n form the q-deformed $su_q(2)$ Lie–Poisson algebra—for by $\{\cdot\}$ we mean $\{x\} = (q^x - q^{-x})/(2\gamma)$. This algebra has a central element $X_n X_n^* + \{H_n\}^2 = \{S\}^2$. The variables X_n^*, X_n enter (4) via $\chi_n = q^{H_n}X_n$ [3]. The variables β_n, β_n^* (the 'electric fields', see equations (5), (6) above) satisfy the Lie–Poisson q-boson algebra

$$\{\beta_n, \beta_m^*\} = \mathrm{i}q^{2N_n}\delta_{mn} \qquad \{N_n, \beta_m\} = -\mathrm{i}\beta_n\delta_{mn}. \tag{9}$$

The quantum counterpart of the LMB system equations (7) is the lattice system described by the quantum Hamiltonian

$$\mathcal{H}_q^L = \frac{1}{2}\sum_{n=1}^{M} J(q)\big[\chi_n^+(\beta_n+\beta_{n+1}) + \chi_n^-(\beta_n^\dagger+\beta_{n+1}^\dagger)\big] - \mathrm{i}q^{2H_n}(\beta_{n+1}^\dagger\beta_n - \beta_n^\dagger\beta_{n+1}) \tag{10}$$

with $J(q) = \sqrt{q-q^{-1}}$, and we shall again choose here $q = \mathrm{e}^\gamma > 1$. We shall also impose periodic boundary conditions $n+M \equiv n$ (the number of lattice sites is even for simplicity). The annihilation and creation operators β_n, $\beta_n^\dagger$ in (10), together with the related number operators $N_n = N_n^\dagger$, form a q-boson algebra with the commutation relations

$$[\beta_n, \beta_m^\dagger] = q^{2N_n}\delta_{mn} \qquad [N_n, \beta_m^\dagger] = \beta_n^\dagger\delta_{mn} \qquad \beta_n = (\beta_n^\dagger)^\dagger. \tag{11}$$

The q-spin operators in (10) are defined as $\chi_n^- = q^{H_n}X_n^-$, $\chi_n^+ = X_n^+q^{H_n}$, where $X_n^\pm$ and H_n are quantum operators belonging to the irreducible $(2S+1)$-dimensional representation of the quantum $su_q(2)$, i.e. $U_q[su(2)]$, algebra. They satisfy

$$[X_n^+, X_m^-] = [2H_n]\delta_{mn} \qquad [H_n, X_m^\pm] = \pm X_n^\pm\delta_{mn}. \tag{12}$$

Here $[\cdot,\cdot]$ denotes the usual commutator, while the 'box' notation $[\cdot]$ means the operation $[x] = (q^x - q^{-x})/(q-q^{-1})$.

Notice that the same continuum limit relations equations (5) apply in this quantum case leading to the physically relevant quantum MB system, namely

$$\mathcal{H} = +\frac{\mathrm{i}}{2}\int_0^L \mathrm{d}x\,\big[\big(\partial_x\mathcal{E}^\dagger(x)\big)\mathcal{E}(x) - \mathcal{E}^\dagger(x)\partial_x\mathcal{E}(x)\big] - \sqrt{\kappa}\int_0^L (S^+(x)\mathcal{E}(x) + S^-(x)\mathcal{E}^\dagger(x)). \tag{13}$$

For $S = 1/2$ this is the two-level so-called Dicke model considered in [20].

A second quantum lattice model like $\mathcal{H}_q^L$ is the Hamiltonian $\boldsymbol{H}_{\mathrm{II}}$; in this notation we change $\mathcal{H}_q^L$ to $\boldsymbol{H}_{\mathrm{I}}$ simply. Thus $\boldsymbol{H}_{\mathrm{I}}$ is given by (10) and $\boldsymbol{H}_{\mathrm{II}}$ is given by

$$\begin{aligned}\boldsymbol{H}_{\mathrm{II}} = -\frac{1}{2}\sum_{n=1}^{M} q^{2H_n}(\beta_n^\dagger\beta_{n-1} + \beta_n\beta_{n-1}^\dagger) - \mathrm{i}\sqrt{\eta}\big((\beta_n^\dagger - \beta_{n-1}^\dagger)\chi_n^- - \chi_n^+(\beta_n - \beta_{n-1})\big)\\ +2\eta^{-1}(q^{2H_n} - q^{-2S}).\end{aligned} \tag{14}$$

We can express the q-boson operators in terms of the ordinary ones b, $b^\dagger$: $[b, b^\dagger] = 1$,

$$\beta = (\beta^\dagger)^\dagger = \sqrt{\frac{[N+1]}{N+1}}\,b \qquad N = b^\dagger b. \tag{15}$$

The q-spins are realized in terms of the usual $su(2)$ generators S^3, $S^\pm$: $[S^3, S^\pm] = \pm S^\pm$, $[S^+, S^-] = 2S^3$

$$\chi^+ = (\chi^-)^\dagger = S^+\sqrt{\frac{[S^3-S][S^3+S+1]}{(S^3-S)(S^3+S+1)}} \qquad H = S^3 \tag{16}$$

and the 'box' notation $[\cdot]$ is used.

3. The classical and quantum Darboux–Bäcklund transformation

In [3] we obtained the zero-curvature representation of the system (4) which means that we constructed an over-determined linear system for a matrix function $\Psi_n(\zeta, t)$ such that

$$\Psi_{n+1} = L(\zeta|n)\Psi_n \tag{17}$$

$$\partial_t\Psi_n = V(\zeta|n)\Psi_n \tag{18}$$

where

$$V(\zeta|n) = \sum_{j=-2}^{2} \zeta^j V_j(n) \qquad L(\zeta|n) = \frac{q^{-N_n-H_n}}{2\gamma} \sum_{j=-2}^{2} \zeta^j L_j(n). \tag{19}$$

Here

$$V_0(n) = 2\mathrm{i}\gamma(\beta_n s_{n-1}^* + \beta_n^* s_{n-1})\sigma^z \qquad V_{\pm 2} = \mp\tfrac{1}{4}\sigma^z \tag{20}$$

$$V_{+1}(n) = -\frac{\sqrt{2\gamma}}{2}\begin{pmatrix} 0 & \mathrm{i}\beta_n^* \\ s_{n-1} & 0 \end{pmatrix} \qquad V_{-1}(n) = \frac{\sqrt{2\gamma}}{2}\begin{pmatrix} 0 & s_{n-1}^* \\ -\mathrm{i}\beta_n & 0 \end{pmatrix} \tag{21}$$

while

$$L_0(n) = 2\mathrm{i}\gamma\begin{pmatrix} \beta_n s_n^* & 0 \\ 0 & \beta_n^* s_n \end{pmatrix} - q^{2(N_n+H_n)}\sigma^z \qquad L_{\pm 2} = \tfrac{1}{2}(\sigma^z \pm I) \tag{22}$$

$$L_{+1}(n) = \sqrt{2\gamma}\begin{pmatrix} 0 & \mathrm{i}\beta_n^* \\ s_n & 0 \end{pmatrix} \qquad L_{-1}(n) = \sqrt{2\gamma}\begin{pmatrix} 0 & s_n^* \\ -\mathrm{i}\beta_n & 0 \end{pmatrix}. \tag{23}$$

The parameter ζ which exists in $\mathcal{C}$ and which appears in equations (17)–(19) will be thought of as the *spectral parameter*, while in continuum limit, recognizing the further spectral parameter λ introduced below, one can see that because the half-space problem interchanges time t and space x it is (18) which becomes the scattering problem in the usual 2×2 sense (the Zakharov–Shabat–AKNS linear system [21]); $\sigma^{x,y,z}$ are the Pauli matrices. The compatibility condition of the two linear system equations (17), (18) under the isospectral condition $\partial_t \zeta = 0$ is

$$\partial_t L(\zeta|n) + L(\zeta|n)V(\zeta|n) - V(\zeta|n+1)L(\zeta|n) = 0 \tag{24}$$

and this coincides with the classical equations (4), independent of ζ. However, $\zeta = \mathrm{e}^{\mathrm{i}\gamma\lambda}$, $\lambda \in \mathcal{C}$ as it was introduced in [3]; λ is a second 'spectral parameter' and the real axis in the λ-plane is the circle of unit radius in the ζ-plane; λ is the usual spectral parameter for the envelope MB equations derived in continuum limit. Notice that time t is suppressed in equations (17), (18): an explicit time dependence will be indicated only where and when it is needed. Reference to equations (17) and (18) may make plain that the function $\Psi_n(\zeta)$ possesses essential singularities of rank 2 at $\zeta = 0, \infty$. It is also important to notice that the linear equations (17) and (18) are invariant under the transformations

$$\Psi_n(\zeta) \to (-1)^{n-1}\sigma^y \Psi_n^*\left(\frac{1}{\zeta^*}\right)\sigma^y \qquad \Psi_n(\zeta) \to \sigma^z \Psi_n(-\zeta)\sigma^z. \tag{25}$$

We can now turn to the derivation of exact solutions of the LMB system equations (4). For this, as mentioned, we developed a variant of the Darboux–Bäcklund dressing procedure [23] rather then any inverse scattering method [21, 22]. The essence of the dressing procedure is to choose a 'seed' solution of the system equations (4), typically some trivial solution, and construct from it a new solution associated with additional points ζ_ν, $\nu = 1, \ldots, N$ (say) of the discrete spectrum: thus $\det \Psi_n(\zeta_\nu, t) = 0$ [21, 23, 24] for the new solution $\Psi_n(\zeta, t)$.

For initial and boundary conditions observe that for the SIT or envelope MB system the typical experimental situation is the half-space problem: an initial optical pulse enters, supposedly without reflection, from $x < 0$ into the resonant medium $x \geqslant 0$ and here breaks up into background radiation and a sequence of soliton pulses. The corresponding mathematical problem is the Cauchy problem at the point $x = 0$: $\varepsilon(x, t)|_{x=0} = \varepsilon_0(t)$ together with the asymptotic boundary conditions (in t) that, for $x > 0$, $\mathcal{N} \to \mathcal{N}_-$, $\rho \to 0$ as $t \to -\infty$. For the so-called 'attenuator' N_- is the ground state $N_- = -1$ of the inversion density. For the lattice problem we therefore take the half-space problem in which $\beta_n(t)$ and $s_n(t)$ are sufficiently decreasing for $|t| \to \infty$, while $H_n(t) \to H$ such that H corresponds to N_-. In this way we would look for a solution in the half-space $n > 0$, for which it becomes the Cauchy problem specified by the conditions

$$\beta_n(t)|_{n=1} = \beta_1(t) \qquad s_n(t)|_{n=1} = s_1(t) \qquad H_n(t)|_{n=1} = H_1(t). \tag{26}$$

With this as motivation, we reported in [1] exact N-soliton solutions derived by the dressing procedure based on the seed solution

$$\beta_n = 0 \qquad s_n = 0 \qquad H_n = H. \tag{27}$$

The Darboux–Bäcklund program can be implemented for an arbitrary number of points of the discrete spectrum, N. The one-soliton case is that when there is only one point of the discrete spectrum $\zeta_0 = \mathrm{e}^{\gamma_0 + \mathrm{i}\alpha_0}$ (say) and $\gamma_0 < 0$. We then found in [1] the formulae

$$\beta_n(t) = \mathrm{i}\sqrt{\frac{2}{\gamma}}\sinh(2\gamma_0)\frac{\exp \mathrm{i}(\phi(n,t)-\alpha_0)}{\cosh(\psi(n,t)-\gamma_0)} \tag{28}$$

$$s_{n-1}(t) = -\sqrt{\frac{2}{\gamma}}\sinh(2\gamma_0)\frac{\exp \mathrm{i}(\phi(n,t)+\alpha_0)}{\cosh(\psi(n,t)+\gamma_0)} \tag{29}$$

$$q^{2(N_n+H_n)} = q^{2H}\frac{1-\tanh(\psi(n,t)-\gamma_0)\tanh\vartheta_0}{1-\tanh(\psi(n,t)+\gamma_0)\tanh\vartheta_0}. \tag{30}$$

Here

$$\phi(n,t) = t\cosh(2\gamma_0)\sin(2\alpha_0) - n\varrho_0 + \phi_0 \tag{31}$$

$$\psi(n,t) = t\sinh(2\gamma_0)\cos(2\alpha_0) - n\vartheta_0 + \psi_0 \tag{32}$$

$$\vartheta_0 = \frac{1}{2}\ln\frac{\sinh^2(\gamma_0 - H\gamma) + \sin^2\alpha_0}{\sinh^2(\gamma_0 + H\gamma) + \sin^2\alpha_0} + 2\gamma_0 \tag{33}$$

$$\varrho_0 = \arg\frac{\sinh(\gamma_0 - H\gamma + \mathrm{i}\alpha_0)}{\sinh(\gamma_0 + H\gamma + \mathrm{i}\alpha_0)} + 2\alpha_0 \tag{34}$$

and ϕ_0 and ψ_0 are arbitrary real constants.

We now turn to the quantum case and discuss the quantum counterpart of the linear system equations (17), (18). In order to solve the two quantum models $\boldsymbol{H}_{\mathrm{I,II}}$, the two models (10), (14), we have to construct the auxiliary *composite model* which may be solved for its eigenstates and eigenvalues by the QISM. We shall show now that the Hamiltonians (10), (14) commute with the generating function of the integrals of motion of this *composite model* and thus are also integrable.

The L-operator of the quantum composite model is defined as

$$L(n|u) = L_S(n|u)L_{QB}(n|u) \tag{35}$$

with the q-boson L-operator equal to

$$L_{QB}(n|u) = \begin{pmatrix} u & \mathrm{i}\sqrt{\eta}\beta_n^\dagger \\ -\mathrm{i}\sqrt{\eta}\beta_n & -u^{-1} \end{pmatrix} \tag{36}$$

and the spin L-operator

$$L_S(n|u) = \begin{pmatrix} u - u^{-1}q^{2H_n} & -\eta\chi_n^+ \\ \eta\chi_n^- & u^{-1} - uq^{2H_n} \end{pmatrix}. \tag{37}$$

Here and in (36) $u \in \mathcal{C}$ is now the spectral parameter and u^* is its conjugate. Further, we shall make use of the following symmetry properties of the composite L-operator (35):

$$L^*(n|u^*) = -\sigma^2 L(n|u^{-1})\sigma^2 \tag{38}$$

$$\mathrm{e}^{\beta(N_n+H_n)}L(n|u)\mathrm{e}^{\frac{1}{2}\beta\sigma^3} = \mathrm{e}^{\frac{1}{2}\beta\sigma^3}L(n|u)\mathrm{e}^{\beta(N_n+H_n)} \tag{39}$$

where β is a complex number, and the star means Hermitian conjugation of the matrix elements without the transposition of the matrix; the σ^i are the Pauli matrices.

The L-operator (35) satisfies the bilinear intertwining relation

$$R(u,v)L(n|u)\otimes L(n|v) = L(n|v)\otimes L(n|u)R(u,v) \tag{40}$$

with the R-matrix

$$R(u,v) = \begin{pmatrix} f(v,u) & 0 & 0 & 0 \\ 0 & g(v,u) & q^{-1} & 0 \\ 0 & q & g(v,u) & 0 \\ 0 & 0 & 0 & f(v,u) \end{pmatrix}$$
$$f(v,u) = \frac{u^2q - v^2q^{-1}}{u^2 - v^2} \qquad g(v,u) = \frac{uv}{u^2 - v^2}(q - q^{-1}) \tag{41}$$

and $u, v \in \mathcal{C}$ spectral parameters.

The monodromy matrix $T(u)$ is introduced as

$$T(u) = L(M|u)L(M-1|u)\cdots L(1|u) = \begin{pmatrix} \boldsymbol{A}(u) & \boldsymbol{B}(u) \\ \boldsymbol{C}(u) & \boldsymbol{D}(u) \end{pmatrix}. \tag{42}$$

The commutation relations of its matrix elements are given by

$$R(u,v)T(u)\otimes T(v) = T(v)\otimes T(u)R(u,v). \tag{43}$$

Under periodic boundary conditions the important quantity is the transfer matrix

$$\tau(u) = \mathrm{Tr}\, T(u) = \boldsymbol{A}(u) + \boldsymbol{D}(u) \tag{44}$$

satisfying

$$[\tau(u), \tau(v)] = 0 \tag{45}$$

for arbitrary $u, v \in \mathcal{C}$.

From (38) and (45) it follows that

$$\tau^\dagger(u^*) = \tau(u^{-1}) \qquad [\tau(u), \tau^\dagger(u)] = 0. \tag{46}$$

The commutativity of the total number operator

$$\boldsymbol{N} = \sum_{n=1}^{M}(N_n + H_n + S) \tag{47}$$

with the transfer matrix $\tau(u)$

$$[\tau(u), \boldsymbol{N}] = 0 \tag{48}$$

is the consequence of the relation (39). From the same relation we can derive the equality

$$\boldsymbol{N}\boldsymbol{B}(u) = \boldsymbol{B}(u)(\boldsymbol{N}+1). \tag{49}$$

Hence, we can consider $\boldsymbol{B}(u)$ as the creation operator of a quasi-particle. By direct calculation it is verified that the total number operator (47) commutes with the Hamiltonians (10) and (14):

$$[\boldsymbol{H}_{\mathrm{I,II}}, \boldsymbol{N}] = 0. \tag{50}$$

It may be proved that the Hamiltonians (10) and (14) commute with the transfer matrix (44):

$$[\boldsymbol{H}_{\mathrm{I,II}}, \tau(u)] = 0. \tag{51}$$

Namely, one can introduce

$$h = \sum_{n=1}^{M} h_{n,n-1} \tag{52}$$

where

$$h_{n,n-1} = \mathrm{i}q^{2H_n}(1 + \eta\beta_n^\dagger\beta_{n-1}) + \eta^{3/2}(\beta_n^\dagger\chi_n^- + \chi_n^+\beta_{n-1}) - \mathrm{i}q^{-2S}. \tag{53}$$

It then follows from these expressions that the operator h (52), with the densities $h_{n,n-1}$, equation (53), and its conjugate $h^\dagger$ commute with the transfer matrix (44)

$$[h, \tau(u)] = [h^\dagger, \tau(u)] = 0. \tag{54}$$

The Hamiltonians (10) and (14) are then expressed through h and $h^\dagger$:

$$2\eta\boldsymbol{H}_{\mathrm{I}} = h + h^\dagger \tag{55}$$

$$2\eta\boldsymbol{H}_{\mathrm{II}} = \mathrm{i}(h - h^\dagger) \tag{56}$$

and commute with $\tau(u)$.

The quantum space of the model is the tensor product of the Fock space and the space of the representation of $U_q[su(2)]$. The local vacuum vector on each lattice site n is given by $|0\rangle_n \otimes |S, -S\rangle_n$, where $|0\rangle_n$ is the vacuum vector: $\beta_n|0\rangle_n = 0$, and $|S, -S\rangle_n$ is the lowest vector of the $(2S+1)$-dimensional representation of $U_q[su(2)]$: $\chi_n^-|S, -S\rangle_n = 0$. The local vacuum vector satisfies the equation

$$\boldsymbol{L}(n|u)|0\rangle_n \otimes |S, -S\rangle_n = \begin{pmatrix} a(u) & * \\ 0 & d(u) \end{pmatrix} |0\rangle_n \otimes |S, -S\rangle_n \tag{57}$$

where the vacuum eigenvalues

$$a(u) = u(u - u^{-1}q^{-2S}) \qquad d(u) = -u^{-1}(u^{-1} - uq^{-2S}). \tag{58}$$

It follows from (57) that the vacuum vector

$$|\Omega\rangle = \prod_{n=1}^{M} |0\rangle_n \otimes |S, -S\rangle_n \tag{59}$$

is an eigenvector of the transfer matrix (44):

$$\tau(u)|\Omega\rangle = (a^N(u) + d^N(u))|\Omega\rangle.$$

The N-particle eigenfunctions of the transfer matrix are constructed in the usual way by applying the creation operator $\boldsymbol{B}(u)$ (42) with (49) to the vacuum vector (59):

$$|\Psi_N(u_1, u_2, \ldots, u_N)\rangle = \prod_{n=1}^{N} \boldsymbol{B}(u_n)|\Omega\rangle. \tag{60}$$

The parameters u_j have to satisfy the Bethe equations

$$\left(\frac{a(u_n)}{d(u_n)}\right)^M = \prod_{\substack{m=1 \\ m\neq n}}^{N} \frac{f(u_m, u_n)}{f(u_n, u_m)}. \tag{61}$$

The eigenvalues $\theta_N(u)$ of $\tau(u)$ to which the eigenstates (60) belong are

$$\tau(u)|\Psi_N\rangle = \theta_N(u)|\Psi_N\rangle$$

$$q^N\theta_N(v) = a^M(v)\prod_{n=1}^{N} f(u_n, v) + d^M(v)\prod_{n=1}^{N} f(v, u_n).$$

The eigenfunction (60) is an eigenfunction of the total number operator (47)

$$\boldsymbol{N}|\Psi_N\rangle = N|\Psi_N\rangle. \tag{62}$$

By substituting (41) and (58) into (61) we can obtain the Bethe equations in the explicit form

$$\left(u_n^2 \frac{u_n q^S - u_n^{-1} q^{-S}}{u_n q^{-S} - u_n^{-1} q^S}\right)^M = \prod_{\substack{m=1\\ m\neq n}}^{N} \frac{u_n^2 q - u_m^2 q^{-1}}{u_n^2 q^{-1} - u_m^2 q}. \tag{63}$$

Thus, after the substitution $u = \mathrm{e}^{\mathrm{i}\gamma\lambda}$ the Bethe equations take the form

$$\mathrm{e}^{-\mathrm{i}p(\lambda_n)M} = \prod_{\substack{m=1\\ m\neq n}}^{N} \frac{\sin(\lambda_n - \lambda_m - \mathrm{i})}{\sin(\lambda_n - \lambda_m + \mathrm{i})} \tag{64}$$

where $p(\lambda)$ is the one-particle momentum

$$p(\lambda) = \mathrm{i} \ln \mathrm{e}^{2\mathrm{i}\gamma\lambda} \left(\frac{\sin\gamma(\lambda - \mathrm{i}S)}{\sin\gamma(\lambda + \mathrm{i}S)}\right). \tag{65}$$

It is appropriate to comment here on the meaning of the *quantum* Darboux–Bäcklund transformation. In the quantum case the linear problem equation (17) must be an operator relation, while the generating function of the quantum V operators is

$$M_n(u, v) = \mathrm{tr}_1(T^+(n|v) \otimes I) R(u - v)(T^-(n|v)). \tag{66}$$

Here tr_1 is the trace with respect to the first space in the tensor product and the $T^{\pm}(\lambda|n)$ are matrix functions with operator values, namely

$$\begin{aligned} T^+(n|v) &= L(M|v) \cdots L(n|v) \\ &\equiv \begin{pmatrix} a^+(n|v) & b^+(n|v) \\ c^+(n|v) & d^+(n|v) \end{pmatrix} \end{aligned} \tag{67}$$

$$\begin{aligned} T^-(n|v) &= L(n|v) \cdots L(1|v) \\ &\equiv \begin{pmatrix} a^-(n|v) & b^-(n|v) \\ c^-(n|v) & d^-(n|v) \end{pmatrix}. \end{aligned} \tag{68}$$

In [25] it was demonstrated that the quantum analogue of the Darboux–Bäcklund transformation is in fact the application of the creation operator of the collective excitation, $\boldsymbol{B}(u)$, to the given state of the quantum system.

The eigenenergies of the Hamiltonians $\boldsymbol{H}_{\mathrm{I,II}}$ are

$$\boldsymbol{H}_{\mathrm{I,II}} |\Psi_N\rangle = E_N^{\mathrm{I,II}} |\Psi_N\rangle$$

with

$$\begin{aligned} E_N^{\mathrm{I}} &= -\sum_{n=1}^{N} \frac{u_n - u_n^2}{2\mathrm{i}} = -\sum_{n=1}^{N} \sin(2\gamma\lambda_n) \\ E_N^{\mathrm{II}} &= -\sum_{n=1}^{N} \frac{u_n + u_n^2}{2} = -\sum_{n=1}^{N} \cos(2\gamma\lambda_n). \end{aligned} \tag{69}$$

The parameters u_n and λ_n are the roots of the Bethe equations (63) and (64) respectively.

Because of the commutativity (50) we can consider

$$\tilde{\boldsymbol{H}}_{\mathrm{II}} = \boldsymbol{H}_{\mathrm{II}} + \boldsymbol{N} \tag{70}$$

as the Hamiltonian of the 'II' model with its eigenenergy. Because of (62) this eigenenergy is

$$E_N^{\mathrm{II}} = \sum_{n=1}^{N} (1 - \cos(2\gamma\lambda_n)) = \sum_{n=1}^{N} 2\sin^2(\gamma\lambda_n). \tag{71}$$

The Bethe equations (64) possess solutions with both real $\lambda_n \in \mathcal{R}$ and complex $\lambda_n^{(k)} = \alpha_n - \frac{\mathrm{i}}{2}(k+1-2m) + \mathcal{O}(\mathrm{e}^{-M})$; $m = 1, \ldots, k$; $\mathrm{Im}\,\alpha_n = 0$. The real solutions correspond to elementary excitations; the complex ones correspond to the m-string solutions (and these are *bound states* of quasi-particles). The existence of the complex-valued solutions $\lambda_n^{(k)}$ indicates the existence of these bound state m-string solutions in both of the models discussed in this paper.

3.1. Continuum limits

The results for the eigenstates and eigenenergies (71) must complete this paper. We have solved the quantum models $\boldsymbol{H}_{\mathrm{I}}$ and $\boldsymbol{H}_{\mathrm{II}}$ (at this non-dynamical level of solution) and shown that these are N-string solutions. Later we hope to show that in matrix elements like (1) with $N \to \infty$ we can regain the classical q-soliton solution (28)–(33) while more generally we hope to find the q-multisoliton solutions of [1]. Here, however, we conclude with continuum limits for each of the two quantum models, $\boldsymbol{H}_{\mathrm{I}}$ and $\boldsymbol{H}_{\mathrm{II}}$.

We consider a discrete medium (of spins) and a continuum of bosons in the limit as the lattice spacing Δ tends to zero. Thus, we consider with $\Delta \to 0$ the case of the 'discrete' medium

$$L = M\Delta \qquad x = n\Delta \qquad \gamma = \frac{c\Delta}{2}$$

$$\beta_n = \sqrt{\Delta}B(x) \qquad B(x) = b(x) - \frac{\Delta\gamma}{2}b^{\dagger}(x)b(x)b(x) + \mathcal{O}(\Delta^2)$$

$$\chi_n^- = \left(1 + \frac{c\Delta}{4}(2S_n^3 + 1)\right)S_n^- + \mathcal{O}(\Delta^2) \qquad H_n = S_n^3.$$

Here the S_n^i are the spin variables (16) and $[b(x), b^{\dagger}(y)] = \delta(x - y)$. In this limit (10) becomes the Hamiltonian of the continuum-limit Dicke model with the integrable 'extended Dicke model': $\boldsymbol{H}_{\mathrm{I}} \to \Delta\boldsymbol{H}_{\mathrm{D}}$,

$$\boldsymbol{H}_{\mathrm{D}} = -\mathrm{i}\int_0^L \mathrm{d}x\, b^{\dagger}(x)\partial_x b(x) - \sqrt{c}\sum_{n=1}^{M}\{b^{\dagger}(x_n)S_n^- + S_n^+ b(x_n)\}. \tag{72}$$

(Note that this usage of Dicke model is that of [20] and *not* that of [10]). For $S = \frac{1}{2}$ the model defined by the Hamiltonian (70) in the 'discrete' medium limit is $\boldsymbol{H}_{\mathrm{II}} \to \Delta^2\boldsymbol{H}$,

$$\boldsymbol{H} = \tfrac{1}{2}\boldsymbol{H}_{\mathrm{BG}} - \sum_{n=1}^{M} S_n^3 b^{\dagger}(x_n)b(x_n) + \mathrm{i}\sqrt{c}\{S_n^+\delta b(x_n) - S_n^-\delta b^{\dagger}(x_n)\} - \tfrac{1}{12}c^2(S_n^3 + \tfrac{1}{2}). \tag{73}$$

Here $\boldsymbol{H}_{\mathrm{BG}}$ is the Hamiltonian of the Bose gas

$$\boldsymbol{H}_{\mathrm{BG}} = \int_0^L \mathrm{d}x\, \partial_x b^{\dagger}(x)\partial_x b(x) + c b^{\dagger}(x)b^{\dagger}(x)b(x)b(x)$$

and

$$\delta b(x_n) = \frac{b(x_n) - b(x_{n-1})}{\Delta}.$$

The Bethe equations (61) for the 'discrete' medium limit (up to the replacement of λ by $\lambda/2$) take the form

$$\mathrm{e}^{-\mathrm{i}\lambda_n L}\left(\frac{\lambda_n - \mathrm{i}cS}{\lambda_n + \mathrm{i}cS}\right)^M = \prod_{\substack{m=1\\ m\neq n}}^{N}\frac{\lambda_n - \lambda_m - \mathrm{i}c}{\lambda_n - \lambda_m + \mathrm{i}c}. \tag{74}$$

The N-particle energy of the Hamiltonian (72) is given by

$$E_{\mathrm{D}} = -\sum_{n=1}^{N}\lambda_n \tag{75}$$

while the N-particle energy of the Hamiltonian (73) is

$$E = \frac{1}{2}\sum_{n=1}^{N}\lambda_n^2. \tag{76}$$

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